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RISE OF AN AIR BUBBLE (A THERMAL) IN THE ATMOSPHERE

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We elucidate the behavior of large regions of warm air (thermals) rising in the earth's atmosphere. It is found that inside a uniformly rising large spherical bubble rotational motion of the air arises around a central vortex line located in the equatorial plane of the bubble. We determine the total energy of the rise process of a thermal.

The rise of large bubbles (thermals) is one of the most common forms of convective motion in the terrestrial atmosphere [1-4]. Such a bubble contains warmer air compared to the surrounding atmosphere, and therefore an Archimedes force acts on it and causes its rise. This process is accompanied by compensating motion of neighboring layers of the atmosphere downward. It is clear that between the surface of the rising bubble and the descending layers of the surrounding atmosphere the forces of viscous friction appear whose magnitude increases with increase in the velocity of the bubble rise. Therefore, the friction forces turn out to be equal to the Archimedes forces already in the initial stage of the process of bubble rise, and the motion of the bubble becomes uniform, so that the medium is considered to be ideal.

We consider a three-dimensional idealized model of a thermal.

Since the air density μ inside the bubble is not much lower than the density of the atmosphere at the same height, we approximately assume the medium to be incompressible and the densities inside and outside the bubble to be equal.

The motion of the air bubble, whose shape is assumed to be spherical, will be interpreted as a local process [5-7]. We consider the bubble itself to be the core of a process in which the air of the bubble does not mix with the surrounding air.

It is known [8] that when a solid sphere moves in a fluid, an excess pressure appears on its faces A and B, and, conversely, a deficit of pressure in its equatorial (middle) plane. This gives rise to forces that strive to deform the sphere, i.e., to compress it in the vertical direction and stretch it equatorially. However, this deformation does not occur because of the rigidity of the sphere.

The situation is different when an air sphere moves in the atmosphere. However, as is seen from experiments, no deformation occurs in this case too. The natural question suggests itself: what hinders the deformation of a rising air bubble? We can obtain an answer to this question if we take into account the fact that the boundary spherical layer of the bubble air, due to its motion relative to the external region of flow, experiences a friction force that generates a torque causing sliding of the boundary spherical layer in a direction opposite to the rise of the bubble. In exactly the same way, the friction forces between the boundary spherical layer and a deeper spherical layer generates similar sliding of this latter layer. This is the way in which deeper and deeper air layers inside the bubble are drawn into internal motion. As a result, all the air in the bubble starts to rotate: layers far from the polar axis AB move in the direction of the external air flow past the bubble (i.e., downward), and layers near the axis in the bubble move along the direction of bubble rise (i.e., upward) (Fig. 2).

This internal rotational motion of the air in the bubble opposes the forces that strive to deform it. Thus, the air of the bubble participates simultaneously in two motions: translational upward and rotational inside the sphere.

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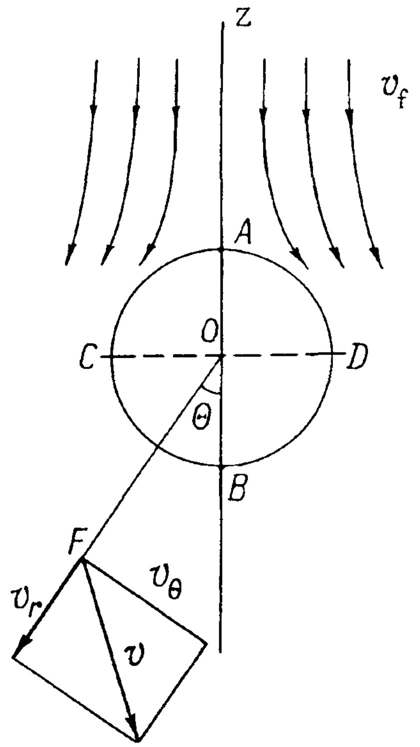

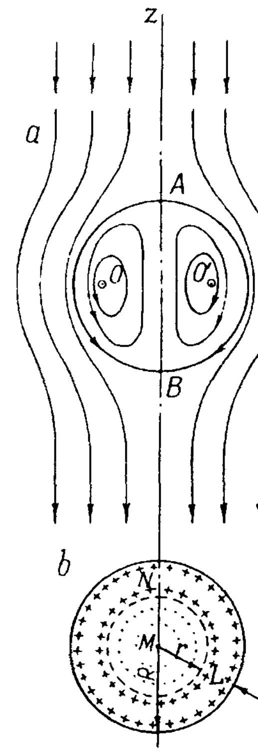

Fig. 1. Motion of a solid sphere in a fluid.

Fig. 2. Streamlines for an air bubble moving in the atmosphere: a) vertical section through the axis; b) middle plane.

To make the picture of the rotational motion in the bubble clearer, we divide its circular equatorial cross section (whose radius is R) by a dashed circle into two parts with equal areas: a disk M, whose radius is $R/\sqrt{2}$, at the axis and a plane ring N, whose width is $R - r$. The areas of the both parts of the equatorial circle are equal to $\pi R^2/2$. In the ring N the air moves downward (in a coordinate system associated with the rising bubble), and in the disk M the lines of rotational motion are directed upward.

The circle L (represented in Fig. 2b by the dashed line) lying in the middle plane and dividing the regions M and N will be called the central vortex line. The air in the thermal rotates around this line. The remaining vortex lines are also circles with centers on the polar axis; they are located in all the planes perpendicular to the axis AB, including the horizontal middle plane. From the symmetry of the problem it is clear that on each circular vortex line rot **v** has the same value.

Thus, the air rotating in the bubble forms a closed vortex tube whose cross section is in the form of a semicircle (Fig. 2) and that curls around the polar axis without a gap and occupies the entire volume of the bubble.

It is only necessary to keep in mind that the surface layer of the air in the bubble comes into contact with a zone of external flow that moves without vortices. Therefore, it follows from the continuity condition that a monotonic decrease in |rot **v**| should occur inside the bubble from the central vortex line to its surface.

We encounter a similar situation in the axial region, where all the rotating streams merge into one rectilinear flow directed upward along the axis AB. This means that in the axial region, too, the angular velocity of air rotation decreases with distance from the central vortex line to the axis.

From what has been said it follows that inside the bubble the quantity |rot **v**| is maximum in the vicinity of the central vortex line L and gradually decreases to zero with distance from the line to each side.

Now, we consider this problem quantitatively. For this purpose, we use well-known solutions for spatial flow around a sphere [8, 9].

In a fixed absolute reference frame the air bubble moves upward along the z axis with the velocity v_0. In a frame of reference moving with the bubble, the surrounding air moves with the velocity $v_f = -v_0$. When a sphere of radius R is immersed in a flow directed along the z axis with the velocity v_f, the stream function ψ and the velocity potential φ far from the bubble are respectively equal to [8, 9]

$$\psi = \frac{1}{2} v_f r^2 \left[1 - \frac{R^3}{r^3}\right] \sin^2 \Theta , \tag{1}$$

$$\varphi = v_f r \left[1 + \frac{1}{2} \frac{R^3}{r^3}\right] \cos \Theta , \tag{2}$$

where r is the radius vector of an air-stream point ($r > R$) drawn from the sphere center, Θ is the angle between the radius vector r and the z axis.

As is known [8, 9], the values of the radial and tangential velocity in the moving coordinate system are equal to (Fig. 1)

$$v_r = v_f \left[1 - \frac{R^3}{r^3}\right] \cos \Theta , \tag{3}$$

$$v_\Theta = - v_f \left[1 + \frac{1}{2} \frac{R^3}{r^3}\right] \sin \Theta . \tag{4}$$

From this it follows that on the sphere surface ($r = R$) the boundary condition of impermeability is satisfied:

$$v_R = 0 , \tag{5}$$

and at infinity ($r \to \infty$)

$$v_r = v_f \cos \Theta , \quad v_\Theta = - v_f \sin \Theta ,$$

i.e., the velocity of the homogeneous flow at infinity is equal to v_f and is directed along the z axis in the negative direction (downward). The velocity distribution on the sphere surface is characterized by the equality

$$v_\Theta = \frac{3}{2} v_f \sin \Theta . \tag{6}$$

The distribution of the pressure p over the sphere surface follows from the Bernoulli equation

$$p + \frac{1}{2} \rho v_\Theta^2 = p_\infty + \frac{1}{2} \rho v_f^2 . \tag{7}$$

From Eq. (7) it follows that

$$\Delta p = p - p_\infty = \left(1 - \frac{9}{4} \sin^2 \Theta\right) \frac{1}{2} \rho v_f^2 , \tag{8}$$

The latter equality shows that the sphere is subjected to a compressive force ($\Delta p > 0$) in the longitudinal direction ($\Theta = 0$) and a tensile force ($\Delta p < 0$) in the lateral direction ($\Theta = \pi/2$). These compressive-tensile forces determine the structure of the air bubble.

For further investigation of the rise of the thermal, we use the notion of the scalar moment of the force M, introduced in [6] and equal to the sum of the scalar products of the forces $\mathbf{F}_i$ applied to the system and the radius vectors $\mathbf{r}_i$ of the points of their application:

$$M = \sum_i (\mathbf{F}_i \cdot \mathbf{r}_i) \tag{9}$$

In particular, for the case of two forces equal in magnitude but directed in opposite directions along a thin rod of length l, the scalar moment is equal to

$$M = F/l\,.$$

According to Hooke's law, the elongation of a rod under the action of the force F is equal to

$$\frac{\Delta l}{l} = kF\,, \tag{10}$$

where k is the elastic modulus of the rod material. From relation (10) it follows that

$$\Delta l = kM\,. \tag{10'}$$

As applied to continuum mechanics, the moment M is determined by the equality

$$M = \int_V (\mathbf{f} \cdot \mathbf{r})\, dV\,, \tag{11}$$

where f is the volumetric density of the force, or by the analogous equality

$$M = \int_S (\mathbf{f} \cdot \mathbf{r})\, dS\,, \tag{11'}$$

where f is the density of the force distributed over the surface S.

In an orthogonal coordinate system the quantity M can be represented as the sum of the components along the axes:

$$M = M_x + M_y + M_z\,,$$

where M_x, M_y, and M_z mean the corresponding integrals, for example,

$$M_z = \int_V f_z z dV\,.$$

We can show that for the pressure force this quantity is equal to

$$M_z = -\int_S pz \cos\Theta\, dS\,. \tag{12}$$

Here Θ is the angle between the normal to the area and the z axis.

Having applied the latter formula to the air bubble, we come to the conclusion that the longitudinal component of the scalar moment of the additional pressure force on the sphere is equal to

$$M_z = -\int_S \Delta p\, z \cos\Theta\, dS\,. \tag{13}$$

Taking into account that S is the spherical surface of the bubble, that the surface element dS and the quantity z are respectively equal to

$$dS = R^2 \sin\Theta\, d\Theta\, d\varphi\,, \quad z = R\cos\Theta\,,$$

and that the excess pressure Δp is determined by equality (8), the expanded version of formula (13) will take the form

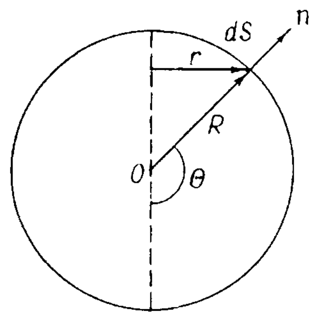

Fig. 3. Diagram of the vectors.

$$M_z = - \int_{\varphi=0}^{2\pi} \int_{\Theta=0}^{\pi} \Delta p \cos \Theta \, R \cos \Theta \, R^2 \sin \Theta \, d\Theta \, d\varphi =$$

$$= 2\pi R^3 \frac{1}{2} \rho v_f^2 \int_0^{\pi} \left(1 - \frac{9}{4} \sin^2 \Theta\right) \cos^2 \Theta \sin \Theta \, d\Theta . \qquad (13')$$

The integral on the right-hand side is easily calculated, yielding for the longitudinal component of the scalar moment the quantity

$$M_z = - 0.1 E_b , \qquad (14)$$

where E_b is the kinetic energy of the rising bubble as a whole, equal to

$$E_b = \frac{4}{3} \pi R^3 \frac{1}{2} \rho v_f^2 . \qquad (15)$$

Now, we calculate the transverse component of the scalar moment:

$$M_h = M_x + M_y = \int_S \Delta p \, (\mathbf{r} \cdot \mathbf{dS}) . \qquad (16)$$

Here $r = \sqrt{x^2 + y^2} = R \sin \Theta$ is the horizontal radius vector of the area dS (whose normal is $\mathbf{n}$, Fig. 3). Since the angle between $\mathbf{r}$ and $\mathbf{n}$ is equal to $\pi/2 - \Theta$, the scalar product $(\mathbf{r} \cdot \mathbf{dS})$ turns out to be equal to $rdS \sin \Theta$. Incorporating the quantities $r = R \sin \Theta$, $dS = R^2 \sin \Theta d\Theta d\varphi$, we come to the relation

$$(\mathbf{r} \cdot \mathbf{dS}) = \sin \Theta \, R \sin \Theta \, R^2 \sin \Theta \, d\Theta \, d\varphi .$$

As a result, formula (16) takes the expanded form

$$M_h = R^3 \int_{\varphi=0}^{2\pi} \int_{\Theta=0}^{\pi} \Delta p \sin^3 \Theta \, d\Theta \, d\varphi . \qquad (17)$$

Integrating over the angle φ and substituting the quantity Δp according to equality (8), we obtain a formula that determines the transverse component of the moment M_h:

$$M_h = 2\pi R^3 \frac{1}{2} \rho v_f^2 \int_0^{\pi} \left(1 - \frac{9}{4} \sin^2 \Theta\right) \sin^3 \Theta \, d\Theta . \qquad (18)$$

Having calculated the integral on the right-hand side, we finally obtain

$$M_h = 1.6 E_b . \qquad (19)$$

From Eqs. (14) and (19) it follows that the core undergoes strong tension in the lateral direction and some compression in the longitudinal direction. The total scalar moment is equal to 1.5 of the motion energy of the core as a single whole:

$$M = 1.5E_b . \tag{20}$$

The value of the moment M is obtained for the pressure from the side of the flow around the bubble. It is clear that the pressure from the side of the core on the external-flow region differs only in direction. Therefore the scalar moment of the pressure force acting on the external-flow region from the side of the core M', just like its components M'_z, M'_h, differs only in sign from the value of the moment calculated above:

$$M' = -M , \quad M'_z = -M_z , \quad M'_h = -M_h .$$

Now, we take into account that according to the general laws of hydrodynamics [8, 9], the energy of the fluid in the region of flow past the bubble E_p is equal to half the energy of the sphere when its density is equal to the fluid density, and the entrained mass is equal to half the mass of the sphere:

$$E_p = 0.5E_b . \tag{21}$$

Now, we go over to determining the velocities and kinetic energies of the air particles participating in the vertical motion of the rising bubble and in the motion around the central vortex line L (Fig. 2). The points OO' on the vertical section of the vortex tube show the wake of the central vortex line.

In the moving coordinate system (fixed in the rising bubble) the air velocity at the periphery of the middle section ($\Theta = \pi/2$) is directed downward and according to formula (6) is equal to $v_h = -3/2v_0$, where v_0 is the velocity of the bubble rise.

The air velocity near the bubble center v_c is numerically equal to v_h, but it is directed upward, and therefore $v_c = 3/2v_0$.

It is clear that in the absolute frame of reference (fixed in the outer atmosphere) the indicated velocities are respectively equal to $v_{ah} = -1/2v_0$, $v_{ac} = 5/2v_0$.

Now, we calculate the total kinetic energy of the air moving in the volume of the bubble V in the moving frame of reference coupled directly to this bubble:

$$E_t = \frac{1}{2} \int_V \rho v^2 dV . \tag{22}$$

where ρ is the air density. Since the air inside the bubble moves along closed stream surfaces (vertical sections of these surfaces are depicted in Fig. 2) and here the velocity components change in a complex manner in going from one stream surface to a neighboring one, we now proceeed as follows.

We determine the relationship between the vertical and horizontal components of the translational velocitiés and the kinetic energies at the peripheral stream surface and at the stream surface adjacent to the axial line OO'.

The streamlines of the air moving inside the bubble are located in planes passing through the coordinate axis z. Near the central vortex line OO^* of the vortex tube the streamlines form circles, and therefore the vertical and horizontal components of the kinetic energy for any such streamline are equal to one another, and, consequently, in the central region of the vortex tube

$$E_z = E_h . \tag{23}$$

The case is more complex for the portions of the peripheral streamlines lying on the surface of the rising bubble, where, according to Eq. (6), the vertical and horizontal projections of the velocity are determined by the equalities

$$v_z = \frac{3}{2} v_f \sin^2 \Theta ; \quad v_h = -\frac{3}{2} v_f \sin \Theta \cos \Theta . \tag{24}$$

The integral kinetic energy of the air particles moving inside the vortex tube over the spherical surface of the bubble is determined by the formula

$$E = \frac{1}{2} \rho \iint_S v^2 \, dS . \tag{25}$$

Taking into account that the area element dS in a spherical coordinate system is equal to $dS = R^2 \sin \Theta d\Theta d\varphi$, we transform equality (25) to

$$E = \frac{1}{2} \rho v_f^2 \int_{\Theta=0}^{\pi} \int_{\varphi=0}^{2\pi} v^2 \sin \Theta \, d\Theta \, d\varphi = \pi \rho R^2 \int_0^{\pi} v^2 \sin \Theta \, d\Theta . \tag{26}$$

Substituting the velocity components (24) into the final expression for E, we obtain formulas for E_z and E_h:

$$\begin{aligned} E_z &= \frac{9}{4} \pi \rho R^2 v_f^2 \int_0^{\pi} \sin^5 \Theta \, d\Theta = \frac{9}{4} \frac{16}{15} \pi R^2 \frac{\rho v_f^2}{2} , \\ E_h &= \frac{9}{4} \pi \rho R^2 v_f^2 \int_0^{\pi} \sin^3 \Theta \cos^2 \Theta \, d\Theta = \frac{9}{4} \frac{4}{15} \pi R^2 \frac{\rho v_f^2}{2} . \end{aligned} \tag{27}$$

Thus, the ratio of the vertical component of the energy E to the horizontal component turns out to be equal to

$$K = \frac{E_z}{E_h} = 4 , \tag{28}$$

whereas, according to (23), in the axial region this ratio is

$$K = 1 . \tag{29}$$

Comparing relations (28) and (29), we can approximately consider that these are the extreme values and that on the average over the cross section of the vortex tube

$$K = 2 \div 3 . \tag{30}$$

As shown in [5, 6], in the absence of a source-sink, the force equilibrium of the aerodynamic region involved in a process that is symmetric about a certain axis (for example, z) is determined by the relations

$$M_z' + P_d + 2E_z = 0 , \tag{31}$$

$$M_h' + 2P_d + 2E_h = 0 . \tag{32}$$

Here $P_d = \int_V (p - p_0) dV$ is the integral difference pressure, M_z' and M_h' are components of the scalar moment of the pressure force acting on the outer surface from within the sphere. We recall that according to (11') $M_z' = -M_z$, $M_h' = -M_h$.

In order to eliminate the unknown quantity P_d, we multiply (31) by 2 and subtract (32). As a result, we obtain $4E_z - 2E_h = 2M_z' - M_h'$. According to equalities (14) and (19), $2M_z - M_h = -1.8E_b$. Consequently,

$$4E_z - 2E_h = 1.8E_b. \tag{33}$$

In particular, assuming $K = 2$, we find for the total energy of the motion of the air inside the bubble $E_t = E_z + E_h = 0.9E_b$. We similarly obtain $E_t = E_z + E_h = 0.72E_b$ for the case $K = 3$. As a result, we conclude that the energy inside the bubble E_t lies within the limits $(0.7-0.9)E_b$.

Summing this value of E_t with the energy of the spherical bubble as a whole E_b and the air energy in the region of flow past the bubble E_p (21), we arrive at the final expression for the integral energy of the air-bubble rise process:

$$\Xi = E_b + E_t + E_p = (2.2 \div 2.4)\, E_b. \tag{34}$$

Thus, it turns out that the total energy of the bubble rise process is 2.2–2.4 times higher than E_b. The entrained mass is equal to 1.2–1.4 of the proper mass of the bubble.

It should be noted that in [10] the so-called Hill spherical vortex is considered, whose stream function inside a sphere of radius R is equal to

$$\psi = -\frac{3v_f}{4R^2}(R^2 - r^2)\, r^2 \sin^2\Theta. \tag{35}$$

The corresponding calculation for this vortex leads to the result $4E_z - 2E_h = 1.3E_b$, which contradicts to the value (33) obtained above. Consequently, the Hill vortex cannot be in force equilibrium in the process of the rise of a thermal, and therefore the opposite assertion made in [9] is erroneous.

In conclusion we emphasize that, as follows from the above, the entrained mass is equal to 1.2–1.4 of the proper mass of the bubble. The large magnitude of the entrained mass reduces the value of the air-bubble velocity noticeably. Neglect of this explains the fact that the velocities calculated earlier (30–50 m/sec) for the rise of thermals were never measured in practice.

NOTATION

ρ, air density; R, bubble radius; $V = 4/3\pi R^3$, bubble volume; r, radius of the axial line located in the middle plane; v_0, velocity of the air-bubble rise; $v_f = -v_0$, velocity of the homogeneous air flow in a moving coordinate system coupled to the bubble; $\mathbf{v}_a$, air velocity in an absolute coordinate system fixed in the atmosphere as a whole; $\mathbf{v}$, air velocity in a moving coordinate system coupled to the rising bubble; v_r, v_Θ, radial and tangential components of the velocity $\mathbf{v}$ in a spherical moving coordinate system; rot $\mathbf{v}$, velocity curl; ψ, stream function for the velocity $\mathbf{v}$; φ, potential of the velocity $\mathbf{v}$; p, air pressure at an arbitrary point; p_∞, air pressure at infinity; $\Delta p = p - p_\infty$, difference pressure at an arbitrary point; $P_d = \int_V \Delta p dV$, integral difference pressure; $\mathbf{f}$, volumetric density of the force; $\mathbf{f} = -\mathrm{grad}\, p$, volumetric density of the pressure force; $M = \int_V (\mathbf{f}\cdot\mathbf{r})dV$, scalar moment of the force $\mathbf{f}$ distributed over the volume V; $M = M_x + M_y + M_z$; $M_x = \int_V f_x x dV$, $M_y = \int_V f_y y dV$, $M_z = \int_V f_z z dV$, components of the scalar moment of the volumetric force $\mathbf{f}$; $M = \int_S (\mathbf{f}\cdot\mathbf{r})dS$, scalar moment of the force $\mathbf{f}$ distributed over the surfaces; $M = M_x + M_y + M_z$; $M_x = \int_S f_x x dS$, $M_y = \int_S f_y y dS$, $M_z = \int_S f_z z dS$, components of the scalar moment of the surface force $\mathbf{f}$; M_z, $M_h = M_x + M_y$, longitudinal (vertical) and transverse components of the scalar moment of the pressure force acting on the sphere from the side of the flow region; $M_z' = -M_z$, $M_h' = -M_h$, longitudinal and transverse components of the scalar moment of the pressure force acting on the flow region from the side of the sphere; $E_b = 1/2\rho v_f^2 V$, kinetic energy of the bubble as a whole; $E_t = 1/2\int_V \rho v^2 dV$, kinetic energy of the air in the bubble volume

V in a moving system; $E_t = E_z + E_h$; $E_z = 1/2\int\limits_V \rho v_z^2 dV$, kinetic energy of the vertical wind in a moving system; $E_h = 1/2\int\limits_V \rho v_x^2 dV + 1/2\int\limits_V \rho v_y^2 dV$, kinetic energy of the horizontal wind in a moving system; $K = E_z/E_h$, ratio of the vertical and horizontal components of the wind energy inside the bubble determined in a moving system; $E_p = E_b/2$, kinetic energy in the region of flow past the bubble; $\Xi = E_b + E_t + E_p$, total energy of the process of bubble rise.

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